Brownian Motion - The Brownian Bridge
Part II - Simulating A Random Path

Gary Schurman, MBE, CFA

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In this white paper we will simulate the random path of a Brownian motion via a Brownian bridge. To that end we will work through the following hypothetical problem from Part I...

Our Hypothetical Problem

We are tasked with simulating a Brownian bridge between a start and end point given the following model parameters... [1]

Table 1: Model Parameters

Description	Value
Value of the Brownian motion at time zero	0.00
Value of the Brownian motion at time T	1.35
Time in years (T)	2.00

Task: Simulate a random path between the Brownian motion's start and end points above.

Model Equations

We defined the variable a to be the known value of the Brownian motion at time $t(a)$, the variable b to be the known value of the Brownian motion at time $t(b)$, and the variable X_t to be the random value of the Brownian motion in the time interval $[t(a), t(b)]$. The equations for the mean and variance of X_t from Part I are... [1]

$$X_t \text{ mean} = b + \frac{(a-b)\,(t(b)-t)}{t(b)-t(a)} \text{ ...and... } X_t \text{ variance} = \frac{(t(b)-t)\,(t-t(a))}{t(b)-t(a)} \text{ ...where... } t \in \Big[t(a), t(b)\Big] \tag{1}$$

Using the mean and variance in Equation (1) above the equation for the random value of X_t is...

$$X_t = \text{mean} + \sqrt{variance}\,z \text{ ...where... } z \sim N\Big[0,1\Big] \tag{2}$$

The Answer To Our Hypothetical Problem

Given the random draws in the table below our random path of the Brownian motion is...

Time	**Expected**	**NDRN**	**Iteration**			
in Years	**Value**	**z**	**0**	**1**	**2**	**3**
0.0000	0.0000	–	0.0000	0.0000	0.0000	0.0000
0.2500	0.1688	-1.4629	–	–	–	**-0.1182**
0.5000	0.3375	0.5132	–	–	**0.7980**	0.7980
0.7500	0.5063	-0.9360	–	–	–	**0.6094**
1.0000	0.6750	0.5766	–	**1.0827**	1.0827	1.0827
1.2500	0.8438	-0.9849	–	–	–	**0.4365**
1.5000	1.0125	-1.4594	–	–	**0.4867**	0.4867
1.7500	1.1813	-1.7047	–	–	–	**0.3157**
2.0000	1.3500	–	1.3500	1.3500	1.3500	1.3500

Note: The bolded values in the random path table above are calculated amounts.

Our plan: Calculate the value of the Brownian motion between two known end points. Iterate until the path is filled out. For our random path we will perform three iterations.

Example: Iteration 2, row 7

$$a = 1.0827 \;\Big|\; b = 1.3500 \;\Big|\; t(a) = 1.0000 \;\Big|\; t(b) = 2.0000 \;\Big|\; t = \frac{t(a) + t(b)}{2} = 1.5000 \;\Big|\; z = -1.4594 \tag{3}$$

Using Equations (1) and (3) above the equation for the mean of X_t is...

$$X_t \text{ mean} = 1.3500 + \frac{(1.0827 - 1.3500)\,(2.0000 - 1.5000)}{2.0000 - 1.0000} = 1.2164 \tag{4}$$

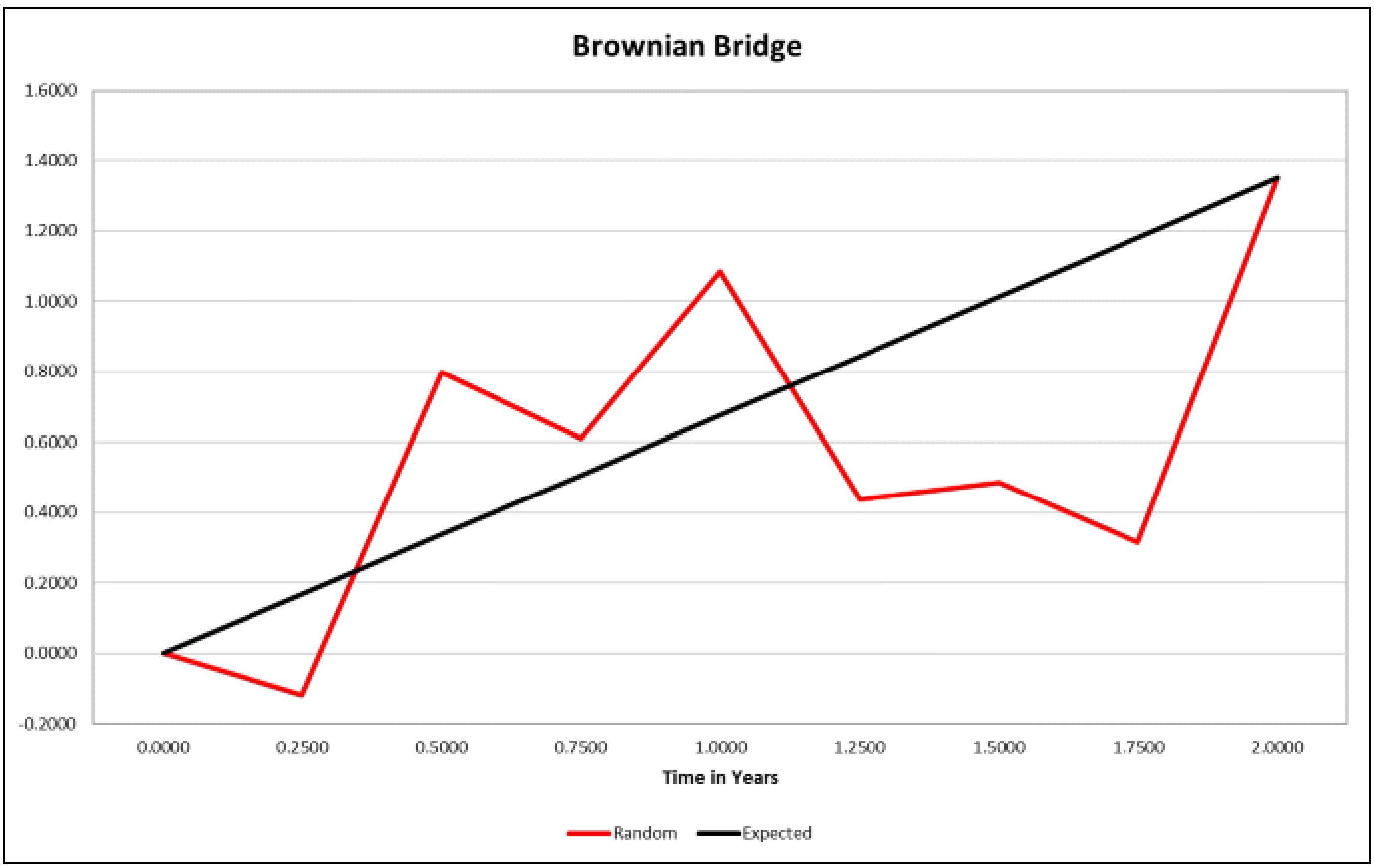
Using Equations (1) and (3) above the equation for the variance of X_t is...

$$X_t \text{ variance} = \frac{(2.0000 - 1.5000)\,(1.5000 - 1.0000)}{2.0000 - 1.0000} = 0.2500 \tag{5}$$

Using Equations (2), (4) and (5) above the value of our path at time $t = 1.5000$ is...

$$X_{1.50} = 1.2164 + \sqrt{0.2500} \times -1.4594 = 0.4867 \tag{6}$$

The graph of our Brownian bridge is...



References

[1] Gary Schurman, *The Brownian Bridge - Base Equations*, August, 2021